Orthogonal polynomials are a class of mathematical fn
Weight fn $w(x)$ in interval $[a, b]$

$$\int_a^b P_n(x) \, P_m(x) \, w(x) \, dx = 0 \quad \text{for } n \neq m$$

The product of any two distinct orthogonal polynomials over the interval $[a, b]$ with respect to $w(x) = 0$

1. Legendre polynomial
2. Hermite polynomial
3. Chebyshev polynomials
4. Jacobi polynomial

we say that the inner product of the functions $f_i(x)$ and $f_j(x)$ is zero. The functions are orthonormal

$$\int_a^b f_i(x) \, f_j(x) \, dx = \delta_{ij} \rightarrow \text{Kronecker delta fn}$$

$$\delta_{ij} = 1 \quad i = j$$
$$= 0 \quad i \neq j$$

Gauss - Legendre quadrature

For integration over the interval $[-1, 1]$ the Gauss - Legendre qu. form approximating the definite integral

$$\int_{-1}^{+1} f(x)\, dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

where $w_i \rightarrow$ weight

$x_i \rightarrow$ nodes, which can be computed using the zeros of the legendre polynomial of degree n.

f^n is symmetric about 0

The abscissas for quadrature order n are given by the roots of Legendre polynomial $P_n(x)$

$n = 3$

$$P_0(X) = 1$$
$$P_1(X) = X$$
$$P_2(X) = \frac{1}{2}(3X^2 - 1)$$
$$P_3(X) = \frac{1}{2}(5X^3 - 3X)$$
$$P_4(x) = \frac{1}{8}(35X^4 - 30X^2 + 3)$$
$$P_5(X) = \frac{1}{8}(63X^5 - 70X^3 + 15x)$$
$$P_6(X) = \frac{1}{16}(231x^6 - 315X^4 + 105X^2 - 5)$$

Powers of L. P.

$$x = P_1(X)$$
$$X^2 = \frac{1}{3}[P_0(X) + 2P_2(X)]$$
$$X^3 = \frac{1}{5}[3P_1(X) + 2P_3(X)]$$
$$\vdots$$

A Closed form for these is given by

$$x^n = \sum_{l=n,\, n=2,\ldots} \frac{(2l+1)\, n!}{2^{(n-1)/2}\left(\frac{1}{2}(n-1)\right)!\,(l+n+1)!!} P_l(x)$$

For interval $(0, 1]$. They obey the orthogonality relationship

$$\int_0^1 \bar{P}_m(X)\, \bar{P}_n(X)\, dX = \frac{1}{2n+1}\, \delta_{mn}$$

$$\bar{P}_0(x) = 1$$
$$\bar{P}_1(x) = 2x - 1$$
$$\bar{P}_2(x) = 6x^2 - 6x + 1$$
$$\bar{P}_3(x) = 20x^3 - 30x^2 + 12x - 1$$

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2}{2n+1}\,\delta_{mn}$$

Rodrigue's formula $\quad P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{\partial x^n} (x^2 - 1)^n$

3 coefficient are $\quad f(x) = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x) + \cdots$

where $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \end{cases}$

For $n = 0$

$$\int_{-1}^{1} f(x)\, P_0(x)\, dx = -\int_{-1}^{0} dx + \int_{0}^{-1} dx = -(0+1) + (1-0) = 0$$

$$\underline{a_0 = 0}$$

For $n = 1$

$$\int_{-1}^{1} f(x)\, P_1(x)\, dx = -\int_{1}^{0} x\, dx + \int_{0}^{1} x\, dx$$

$$= -\frac{x^2}{2}\Big|_{-1}^{0} + \frac{x^2}{2}\Big|_{0}^{1} = 1$$

$$\|P_1\|^2 = \int_{-1}^{1} x^2\, dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\Rightarrow a_1 = 3/2$$

For $n = 2$

$$\int_{-1}^{1} f(x)\, P_2(x)\, dx = -\int_{-1}^{0} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx + \int_{0}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2}\right) dx$$

$$= -\frac{x^3}{2}\Big|_{-1}^{0} + \frac{x^3}{2}\Big|_{0}^{1} + \frac{1}{2}x\Big|_{-1}^{0} - \frac{1}{2}x\Big|_{0}^{1} = 0$$

$$a_3 = 0$$

So $a_1 = 0$, $\quad a_1 = \frac{3}{2}$ $\quad a_2 = 0$

Gauss Quadrature

It turn out that for a given $n+1$ points that y's are the root of $(n+1)$th order Legendre Polynomial.

General equation.

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(X) - \frac{n}{n+1} P_{n-1}(x)$$

$$P_0(X) = 1 \qquad P_1(X) = X$$

$$P_2(X) = \frac{3}{2}\cancel{x}X^2 - \frac{1}{2} = (3x^2-1)\frac{1}{2} = 0$$

$$X = \pm\sqrt{\frac{1}{3}}$$

$$P_3(X) = \frac{5}{2} \cdot X^3 - \frac{3}{2}X = 0$$

$$X = 0, \quad X = \pm\sqrt{3/5}$$

Hence Gauss Quadrature is also called Gauss-Legendre quadrature

Weight $= w = 1$ $\qquad n = 3$ $\qquad$ $P_n(X)$ given in Assignment -6 Question.

$$P_3(X) = \frac{1}{2}(5x^3 - 3x) = 0$$

$$X = 0, \pm\sqrt{\frac{3}{5}}$$

$$X_0 = -\sqrt{\frac{3}{5}} \qquad X_1 = 0; \qquad X_2 = \sqrt{\frac{3}{5}}$$

$$W_0 = \int_{-1}^{1} \frac{(X-X_1)(X-X_2)}{(X_0-X_1)(X_0-X_2)}dx = \frac{5}{6}\int_{-1}^{1} X\left(X-\sqrt{\frac{3}{5}}\right)dx = \frac{5}{9}$$

$$W_1 = \int_{-1}^{1} \frac{(X-X_0)(X-X_2)}{(X_1-X_0)(X_1-X_2)}dx = -\frac{5}{3}\int_{-1}^{1}\left(X+\sqrt{\frac{3}{5}}\right)\left(X-\sqrt{\frac{3}{5}}\right)dx = \frac{8}{9}$$

$$W_2 = \int_{-1}^{1} \frac{(X-X_0)(X-X_1)}{(X_2-X_0)(X_2-X_1)}dx = \frac{5}{6}\int_{-1}^{1} X\left(X+\sqrt{\frac{3}{5}}\right)dx = \frac{5}{9}$$

$W_0 = 0.5555556$

$W_1 = 0.88888889$

$W_2 = 0.5555556$

So for fn

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\sum_{i=0}^{n} a_i \cdot f\left(\frac{(b-a)}{2} x_i + \frac{a+b}{2}\right)$$

weight function

We defined in class that

$$P(X) = Q(X) \cdot P_{n+1}(X) + R(X)$$

$$\int_{-1}^{1} P(X)\,dX = \int_{-1}^{1} Q(X) \cdot P_{n+1}(X) + R(X)\,dX$$

Since the degree Q(X) is less than of $P_{n+1}(x)$ we have

$$\int_{-1}^{1} Q(X)\, P_{n+1}(X)\,dX = 0$$

$$\int_{-1}^{1} P(x)\,dx = \int_{-1}^{1} R(x)\,dx = \sum_{i=0}^{n} R(x_i)$$

$$\equiv \sum_{i=0}^{n} P_{n+1}(X_i)\, Q(X_i) + R(x_i)$$

Implementation of Gaussian Quadrature.

where the function is zero.

$X_0 = -\sqrt{3/5} = 0.34641$

$X_1 = 0 = 0$

$X_2 = +\sqrt{3/5} = -0.34641$

$\{X_1\ X_2\ X_3\}$

Similarly for weight are

$w_0 = 0.555556$

$w_1 = 0.888889$

$w_2 = 0.555556$

$\{w_0, w_1, w_2\}$

Now we can do integration

$$\int_{-1}^{+1} w\, f(x) = \int_{-1}^{1} w_i\, f(x_i)\,dx$$

also we change the limit Rest is in Pro